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As has been pointed out in meetings with MSFC personnel, there is a natural relation between Markov chains, matrix theory, and graph theory. Indeed, Gantmacher [1] derives many results of the theory of finite Markov chains from linear algebra, i.e., the theory of finite matrices. The theory of graphs is hardly new, having been used by Euler in the solution of the famous "Bridges of Königsberg" problem. König [2] presented additional applications of the theory in 1916. However only in this and the preceding decade has the subject begun to receive widespread interest, and the field is yet but imperfectly explored.

One interest, then, in this next quarter, will be a retracing of the above-mentioned derivation of Gantmacher making use of graph-theoretic methods, and carrying out such extensions of this derivation as are of interest in the theory of finite Markov chains. Some attention will be given to the question of the reducibility and imprimitivity of matrices of large order. A second objective, time permitting, will be an attempt to apply the theory of infinite matrices and graphs to the study of infinite Markov chains. The reader will appreciate the different character of this second objective on recalling that whereas the finite matrix is of interest to linear algebra, the study of infinite matrices belongs to the discipline of analysis, and that the theory of infinite graphs is an aspect of topology.

The material contained in the present report concludes the presentation of the basic aspects of Markov chains. An oral presentation is planned for September.

In order to study the sums of independent random variable as Markov chains, we introduce the functions

$$ {}_kp_{ij}^{n} = \Pr\{X_n = j;\ X_\nu \neq k;\ \nu = 1, \dots, n-1 \mid X_0 = i\} \qquad (n \geq 1,\ k \neq j)$$

$$ {}_kf_{ij}^{n} = \Pr\{X_n = j;\ X_\nu \neq j, X_\nu \neq k;\ \nu = 1, \dots, n-1 \mid X_0 = i\}$$

where for convenience we take

$$ {}_kp_{ij}^{0} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$ {}_kf_{ij}^{0} \equiv 0$$

The state k is called a taboo state, and the probabilities introduced here are called taboo transition probabilities. A verbal interpretation is obvious. Moreover, it is easily seen that

(*) $f_{ij}^{n} = \sum_{\nu=0}^{n} {}_jp_{ii}^{\nu}\ {}_if_{ij}^{n-\nu} \quad (j \neq i)$

(**) ${}_kp_{ij}^{n} = \sum_{\nu=0}^{n} {}_kf_{ij}^{\nu}\ {}_kp_{jj}^{n-\nu} \quad (k \neq j)$

(***) $p_{ij}^{n} = \sum_{\nu=0}^{n} p_{ii}^{\nu}\ {}_ip_{ij}^{n-\nu} \quad (j \neq i)$

(****) $p_{ij}^{n} = \sum_{\nu=0}^{n} f_{ij}^{\nu}\ p_{jj}^{n-\nu}$

We define the generating functions

$$ {}_kf_{ij}(s) = \sum_{n=0}^{\infty} {}_kf_{ij}^{n} s^n$$

$$ {}_kp_{ij}(s) = \sum_{n=0}^{\infty} {}_kp_{ij}^{n} s^n$$

<u>Lemma 1</u>. If $i \neq j$ then

$$ f_{ij}(s) = {}_jp_{ii}(s)\ {}_if_{ij}(s)$$

<u>Proof</u>:

$$ {}_jp_{ii}(s)\ {}_if_{ij}(s) = \sum_{n=0}^{\infty} \left(\sum_{\nu=0}^{n} {}_jp_{ii}^{\nu}\ {}_if_{ij}^{n-\nu} \right) s^n = \sum_{n=0}^{\infty} f_{ij}^{n} s^n = f_{ij}(s). \quad \blacksquare$$

Recall Abel's lemma: If

(a) $\sum_{k=0}^{\infty} a_k = a < \infty$ then $\lim_{s\to 1-0} \sum_{k=0}^{\infty} a_k s^k = \sum_{k=0}^{\infty} a_k = a$

(b) $a_k \geq 0$ and $\lim_{s\to 1-0} \sum_{k=0}^{\infty} a_k s^k = a \leq \infty$ then $\sum_{k=0}^{\infty} a_k = a$

Lemma 2. If $i \leftrightarrow j$ then

(A) $\lim_{s\to 1-0} {}_jP_{ii}(s) = \left(\sum_{n=1}^{\infty} f_{ij}^n\right) / \left(\sum_{n=1}^{\infty} {}_if_{ij}^n\right) < \infty$

(B) ${}_jP_{ii}^* = \sum_{n=0}^{\infty} {}_jP_{ii}^n = \lim_{s\to 1-0} {}_jP_{ii}(s)$

Proof: Since $\sum_{n=0}^{\infty} f_{ij}^n \leq 1$, $\sum_{n=0}^{\infty} {}_if_{ij}^n \leq 1$

we have by the (a) part of Abel's lemma that

$$\lim_{s\to 1-0} f_{ij}(s) = \lim_{s\to 1-0} \sum_{n=0}^{\infty} f_{ij}^n s^n = \sum_{n=0}^{\infty} f_{ij}^n \leq 1$$

and $\lim_{s\to 1-0} {}_if_{ij}(s) = \lim_{s\to 1-0} \sum_{n=0}^{\infty} {}_if_{ij}^n s^n = \sum_{n=0}^{\infty} {}_if_{ij}^n \leq 1$

Since $i \leftrightarrow j$ $\exists n > 0 \ni f_{ij}^n > 0$, and by (*) this implies that

$${}_if_{ij}^{n-\nu} > 0$$

for ~~some~~ $\nu \ni 0 \leq \nu \leq n-1$. Hence

$$\sum_{n=0}^{\infty} {}_if_{ij}^n > 0$$

Thus, from lemma 1, it follows that

$$\lim_{s\to 1-0} {}_jP_{ii}(s) = \lim_{s\to 1-0} \sum_{n=0}^{\infty} {}_jP_{ii}^n s^n = \frac{\lim_{s\to 1-0} f_{ij}(s)}{\lim_{s\to 1-0} {}_if_{ij}(s)} = \frac{\sum_{n=1}^{\infty} f_{ij}^n}{\sum_{n=1}^{\infty} {}_if_{ij}^n}$$

Then by the (b) part of Abel's lemma,

$${}_jP_{ii}^* = \sum_{n=0}^{\infty} {}_jP_{ii}^n = \lim_{s\to 1-0} {}_jP_{ii}(s) < \infty$$

and this completes the proof.

Lemma 3. If

1° $c_n = \sum_{\nu=0}^{n} a_{n-\nu} b_\nu \qquad (n \geq 0)$

2° $0 \leq a_n \leq K \qquad (n \geq 0)$

3° $\sum_{n=0}^{\infty} a_n = \infty$

4° $\lim_{n\to\infty} b_n = b$

Then

$$\lim_{n\to\infty} \frac{c_n}{\sum_{\nu=0}^{n} a_\nu} = b$$

Proof: Note that

$$|b_n| < M \qquad \forall n \geq 0$$

$$|b_\nu - b| < \varepsilon \qquad \forall \varepsilon > 0 \ . \ \nu \geq N(\varepsilon)$$

$$\sum_{\nu=0}^{n} a_{n-\nu} = \sum_{\nu=0}^{n} a_\nu$$

Then

$$\left| \frac{c_n}{\sum_{\nu=0}^{n} a_\nu} - b \right| = \left| \frac{\sum_{\nu=0}^{n} a_{n-\nu}(b_\nu - b)}{\sum_{\nu=0}^{n} a_\nu} \right| \leq \left| \frac{\sum_{\nu=0}^{N-1} a_{n-\nu}(b_\nu - b)}{\sum_{\nu=0}^{n} a_\nu} \right| + \left| \frac{\sum_{\nu=N}^{n} a_{n-\nu}(b_\nu - b)}{\sum_{\nu=0}^{n} a_\nu} \right|$$

$$< \frac{2MNK}{\sum_{\nu=0}^{n} a_\nu} + \varepsilon \left| \frac{\sum_{\nu=N}^{n} a_{n-\nu}}{\sum_{\nu=0}^{n} a_\nu} \right| < \frac{2MNK}{\sum_{\nu=0}^{n} a_\nu} + \varepsilon$$

$$\therefore \quad \lim_{n \to \infty} \left| \frac{c_n}{\sum_{\nu=0}^{n} a_\nu} - b \right| < \varepsilon$$

Since ε can be taken as small as we please,

$$\lim_{n \to \infty} \left| \frac{c_n}{\sum_{\nu=0}^{n} a_\nu} - b \right| = 0$$

i.e., $\displaystyle \lim_{n \to \infty} \frac{c_n}{\sum_{\nu=0}^{n} a_\nu} = b$. ∎

Theorem 1. Let i and j be arbitrary states such that j is recurrent. Then

$$\lim_{m \to \infty} \frac{\sum_{n=0}^{m} P_{ij}^{n}}{\sum_{n=0}^{m} P_{jj}^{n}} = f_{ij}^{*} = \sum_{n=1}^{\infty} f_{ij}^{n}$$

Proof: We have that

$$\sum_{n=0}^{m} P_{ij}^{n} = \sum_{n=0}^{m} \sum_{\nu=0}^{n} f_{ij}^{\nu} P_{jj}^{n-\nu} = \sum_{n=0}^{m} \sum_{\nu=0}^{n} P_{jj}^{\nu} f_{ij}^{n-\nu} = \sum_{n=0}^{m} \sum_{\nu=0}^{\infty} P_{jj}^{\nu} f_{ij}^{n-\nu}$$

where we take $f_{ij}^{n-\nu} = 0$ for $\nu > n$.

Since each summation is actually finite, we may interchange the order of summation to obtain

$$\sum_{n=0}^{m} P_{ij}^{n} = \sum_{\nu=0}^{\infty} P_{jj}^{\nu} \sum_{n=0}^{m} f_{ij}^{n-\nu} = \sum_{\nu=0}^{\infty} P_{jj}^{\nu} \sum_{n=0}^{m-\nu} f_{ij}^{n}$$

Take

$$F_{ij}^{m} = \begin{cases} \sum_{n=0}^{m} f_{ij}^{n} & , m \geq 0 \\ 0 & , m < 0 \end{cases}$$

So

$$\sum_{n=0}^{m} P_{ij}^{n} = \sum_{\nu=0}^{\infty} P_{jj}^{\nu} F_{ij}^{m-\nu} = \sum_{\nu=0}^{m} P_{jj}^{\nu} F_{ij}^{m-\nu} = \sum_{\nu=0}^{m} P_{jj}^{m-\nu} F_{ij}^{\nu}$$

Recall the

Lemma 3. If

1° $c_n = \sum_{\nu=0}^{n} a_{n-\nu} b_\nu \qquad (n \geq 0)$

2° $0 \leq a_n \leq K \qquad (n \geq 0)$

3° $\sum_{\nu=0}^{\infty} a_\nu = \infty$

4° $\lim_{n\to\infty} b_n = b$

then $$\lim_{n\to\infty} \frac{c_n}{\sum_{\nu=0}^{n} a_\nu} = b$$

Take $$c_m = \sum_{n=0}^{m} p_{ij}^{n} = \sum_{\nu=0}^{m} p_{jj}^{m-\nu} F_{ij}^{\nu}$$
$$a_m = p_{jj}^{m}$$
$$b_n = F_{ij}^{m} = \sum_{n=0}^{m} f_{ij}^{n}$$

Clearly $$0 \leq p_{jj}^{m} \leq 1$$
$$\sum_{n=0}^{\infty} p_{jj}^{n} = \infty \quad (j \text{ is recurrent})$$
$$\lim_{m\to\infty} b_m = \sum_{n=0}^{\infty} f_{ij}^{n} = b \leq 1$$

Then by the lemma $$\lim_{m\to\infty} \frac{\sum_{n=0}^{m} p_{ij}^{n}}{\sum_{n=0}^{m} p_{jj}^{n}} = \sum_{n=0}^{\infty} f_{ij}^{n} = f_{ij}^{*} \quad ∎.$$

Lemma 4. $${}_ip_{ij}(s) = {}_if_{ij}(s) \cdot {}_ip_{jj}(s) \qquad (i \neq j)$$

Proof: $${}_if_{ij}(s)\,{}_ip_{jj}(s) = \sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n} {}_if_{ij}^{\nu}\,{}_ip_{jj}^{n-\nu}\right)s^n = \sum_{n=0}^{\infty} {}_ip_{ij}^{n} s^n = {}_ip_{ij}(s)$$

Lemma 5. If $i \to j$, then
$${}_ip_{ij}^{*} \equiv \sum_{n=0}^{\infty} {}_ip_{ij}^{n} = \lim_{s\to 1-0} {}_ip_{ij}(s) < \infty$$

Proof: By lemma 2, part B
$${}_ip_{jj}^{*} = \sum_{n=0}^{\infty} {}_ip_{jj}^{n} = \lim_{s\to 1-0} {}_ip_{jj}(s) < \infty$$

Also, since $\sum_{n=0}^{\infty} {}_if_{ij}^{n} < \infty$ we have by the (a) part of Abel's lemma that
$$\lim_{s\to 1-0} {}_if_{ij}(s) = \lim_{s\to 1-0} \sum_{n=0}^{\infty} {}_if_{ij}^{n} s^n = \sum_{n=0}^{\infty} {}_if_{ij}^{n} < \infty$$

Now

$$\lim_{s\to 1-0} {}_ip_{ij}(s) = \lim_{s\to 1-0} {}_if_{ij}(s)\cdot {}_ip_{jj}(s) = \sum_{n=0}^{\infty} {}_if_{ij}^{n} \cdot \sum_{n=0}^{\infty} {}_ip_{jj}^{n} < \infty$$

$$= \sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{\infty} {}_if_{ij}^{\nu}\, {}_ip_{jj}^{n-\nu}\right) = {}_ip_{ij}^{*} \qquad \blacksquare.$$

__Theorem 2.__ If i and j are in the same recurrent class, then

$$\lim_{m\to\infty} \frac{\sum_{n=0}^{m} p_{ij}^{n}}{\sum_{n=0}^{m} p_{ii}^{n}} = {}_ip_{ij}^{*}$$

__Proof:__ We have

$$p_{ij}^{n} = \sum_{\nu=0}^{n} p_{ii}^{\nu}\, {}_ip_{ij}^{n-\nu}$$

$$\sum_{n=0}^{m} p_{ij}^{n} = \sum_{n=0}^{m}\sum_{\nu=0}^{n} p_{ii}^{\nu}\, {}_ip_{ij}^{n-\nu} = \sum_{n=0}^{m}\sum_{\nu=0}^{\infty} p_{ii}^{\nu}\, {}_ip_{ij}^{n-\nu}$$

$\because\ {}_ip_{ij}^{n-\nu} = 0$ for $\nu > n$. We may interchange the order of summation, so that $\sum_{n=0}^{m} p_{ij}^{n} = \sum_{\nu=0}^{\infty} p_{ii}^{\nu} \sum_{n=0}^{m} {}_ip_{ij}^{n-\nu} = \sum_{\nu=0}^{\infty} p_{ii}^{\nu} \sum_{n=0}^{m-\nu} {}_ip_{ij}^{n}$

Take

$${}_iP_{ij}^{m} = \begin{cases} \sum_{n=0}^{m} {}_ip_{ij}^{n}, & m \geq 0 \\[2ex] 0 & m < 0 \end{cases}$$

Then $\sum_{n=0}^{m} p_{ij}^{n} = \sum_{\nu=0}^{\infty} p_{ii}^{\nu}\, {}_iP_{ij}^{m-\nu} = \sum_{\nu=0}^{m} p_{ii}^{\nu}\, {}_iP_{ij}^{m-\nu} = \sum_{\nu=0}^{m} p_{ii}^{m-\nu}\, {}_iP_{ij}^{\nu}$

We now apply lemma 3 with

$$c_m = \sum_{n=0}^{m} p_{ij}^{n} = \sum_{\nu=0}^{m} p_{ii}^{m-\nu}\, {}_iP_{ij}^{\nu}$$

$$a_\nu = p_{ii}^{\nu}$$

$$b_\nu = {}_iP_{ij}^{\nu} = \sum_{n=0}^{\nu} {}_ip_{ij}^{n}$$

$$b = \sum_{n=0}^{\infty} {}_ip_{ij}^{n}$$

to obtain the result.

__Remark:__ If $i \leftrightarrow j$ we define the r.v.'s

$$U_n = U(i,j,n) = \begin{cases} 1, & \text{if the process starting in state } i \text{ is in state } j \text{ after } n \text{ steps without having been in state } i \text{ in the interim} \\ 0, & \text{Otherwise} \end{cases}$$

Then $\quad E(U_n) = {}_iP_{ij}^{n}$

and $\quad E\left(\sum_{n=0}^{\infty} U_n\right) = \sum_{n=0}^{\infty} E(U_n) = {}_iP_{ij}^{*}$

Thus it follows from the theorem that ${}_iP_{ij}^*$ is the expected number of visits to state j between successive visits to state i.

__Lemma__ If i and j belong to the same recurrent class, then

$$\lim_{m\to\infty}\left(\sum_{n=0}^{m} P_{jj}^{n} \Big/ \sum_{n=0}^{m} P_{ii}^{n}\right) = {}_iP_{ij}^{*}$$

__Proof:__ $$\lim_{m\to\infty} \frac{\sum_{n=0}^{m} P_{jj}^{n}}{\sum_{n=0}^{m} P_{ii}^{n}} = \lim_{m\to\infty} \frac{\sum_{n=0}^{m} P_{jj}^{n}}{\sum_{n=0}^{m} P_{ij}^{n}} \cdot \frac{\sum_{n=0}^{m} P_{ij}^{n}}{\sum_{n=0}^{m} P_{ii}^{n}} = {}_iP_{ij}^{*} \cdot \frac{1}{f_{ij}^{*}} = {}_iP_{ij}^{*}$$

Since $f_{ij}^* = 1$

__Lemma__ If i and j belong to the same recurrent class,

$${}_iP_{ij}^{*} = {}_iP_{jj}^{*} / {}_jP_{ii}^{*}$$

__Proof:__ It is easily verified that

(*) $\quad f_{ij}(s) = {}_jP_{ii}(s)\ {}_if_{ij}(s)$

(**) $\quad {}_iP_{ij}(s) = {}_if_{ij}(s)\ {}_iP_{jj}(s)$

Applying Abel's Lemma gives

$$f_{ij}^{*} = {}_jP_{ii}^{*}\ {}_if_{ij}^{*}$$

$${}_iP_{ij}^{*} = {}_if_{ij}^{*}\ {}_iP_{jj}^{*}$$

hence, as $f_{ij}^* = 1$, we have

$${}_iP_{ij}^{*} = {}_iP_{jj}^{*} / {}_jP_{ii}^{*}$$

Recall the

__Theorem:__ In a positive recurrent periodic class with states j=0,1,2,...

$$\lim_{n\to\infty} P_{jj}^{n} = \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}\ , \quad \sum_{0}^{\infty} \pi_i = 1$$

and the 's are uniquely determined by

$$\pi_i \geq 0\ , \quad \sum_{i=0}^{\infty} \pi_i = 1\ , \quad \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$$

An immediate consequence of this is the

Corollary: In an irreducible positive recurrent class with states 0, 1, 2,... the stationary distribution $\{d\pi_i\}_0^\infty$ constitutes a convergent positive solution to the system of equations $\sum_{i=0}^{\infty} x_i p_{ij} = x_j \quad (j=0,1,\ldots)$

We shall now show that this property characterizes positive recurrence.

Theorem: Let $\mathfrak{P}$ be an irreducible Markov chain. If the system of equations $\sum_{j=0}^{\infty} x_j p_{ji} = x_i \quad (i=0,1,\ldots)$ has a nontrivial convergent solution $\{x_i\}_0^\infty$ $\left(\text{i.e., } \sum_0^\infty |x_i| < \infty\right)$ then $\mathfrak{P}$ is positive recurrent.

Proof: By simple iteration we obtain

$$\sum_{j=0}^{\infty} x_j p_{ji}^{n} = x_i$$

Since $\left|\sum_{j=0}^{\infty} x_j p_{ji}^{n}\right| \leq \sum_{j=0}^{\infty} |x_j p_{ji}^{n}| \leq \sum_{j=0}^{\infty} |x_j| < \infty$

this series is absolutely convergent. Let $\mathcal{P}_{ji}^{m} = \frac{1}{m}\sum_{n=1}^{m} p_{ji}^{n}$

$$\text{Thus} \quad \frac{1}{m}\sum_{n=1}^{m}\sum_{j=0}^{\infty} x_j p_{ji}^{n} = \sum_{j=0}^{\infty} x_j\left(\frac{1}{m}\sum_{n=1}^{m} p_{ji}^{n}\right) = \sum_{j=0}^{\infty} x_j \mathcal{P}_{ji}^{m} = \frac{1}{m}\sum_{n=1}^{m} x_i = x_i$$

i.e.,

$$\sum_{j=0}^{\infty} x_j \mathcal{P}_{ji}^{m} = x_i$$

The series on the left is absolutely and uniformly convergent by the Weierstrass M-test, so that

$$x_i = \lim_{m\to\infty}\sum_{j=0}^{\infty} x_j \mathcal{P}_{ji}^{m} = \sum_{j=0}^{\infty} x_j \lim_{m\to\infty}\mathcal{P}_{ji}^{m} = \sum_{j=0}^{\infty} x_j \pi_i = \pi_i \sum_{j=0}^{\infty} x_j$$

Since $x_i \neq 0$ for some i and $\sum_{j=0}^{\infty} |x_j| < \infty$ it follows that $\pi_i \neq 0$, i.e., $\pi_i > 0$ so that $\mathfrak{P}$ is positive recurrent.

Theorem: For a recurrent irreducible M.C. the positive sequence given by $v_0 = 1$, $v_i = {}_0p_{0i}^*$ $(i=1,2,\ldots)$

is a solution of the system of equations

$$v_i = \sum_{j=0}^{\infty} v_j p_{ji} \qquad (i=0,1,\ldots)$$

Proof: By the definition of ${}_0p_{0i}^*$

$$\sum_{j=0}^{\infty} v_j p_{ji} = \sum_{j=1}^{\infty} {}_0p_{0j}^* p_{ji} + p_{0i} = p_{0i} + \sum_{j=1}^{\infty} \left(\sum_{n=1}^{\infty} {}_0p_{0j}^{n}\right) p_{ji}$$

If the double series on the right is convergent then, since it contains only non-negative terms, it is absolutely convertent, and we may write

$$(*) \qquad \sum_{j=0}^{\infty} v_j p_{ji} = p_{0i} + \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} {}_0p_{0j}^{n} p_{ji}$$

where it remains to show the convergence of the double series on the right.

Now
$$\sum_{j=1}^{\infty} {}_0p_{0j}^{n} p_{ji} = \begin{cases} {}_0p_{0i}^{n+1}, & i \neq 0 \\ f_{00}^{n+1}, & i = 0 \end{cases}$$

Thus, if $i \neq 0$ we have in place of (*) that

$$(**) \qquad \sum_{j=0}^{\infty} v_j p_{ji} = p_{0i} + \sum_{n=1}^{\infty} {}_0p_{0i}^{n+1} = \sum_{n=0}^{\infty} {}_0p_{0i}^{n+1} = \sum_{n=0}^{\infty} {}_0p_{0i}^{n} = {}_0p_{0i}^{*} = v_i$$

If $i=0$, then in place of (*) we have

$$(***) \qquad \sum_{j=0}^{\infty} v_j p_{ji} = p_{00} + \sum_{n=1}^{\infty} f_{00}^{n+1} = \sum_{n=0}^{\infty} f_{00}^{n} = 1 = v_0$$

Thus (**) and (***) establish the required convergence property, and, in fact, the theorem. ∎

Theorem: For a recurrent irreducible Markov chain, the system

$$(1) \qquad v_i = \sum_{j=0}^{\infty} v_j p_{ji} \qquad (i=0,1,2,\ldots)$$

subject to the conditions

(2) $v_0 = 1 \quad , v_i \geq 0 \quad (i = 1, 2, \ldots)$

has a unique solution.

<u>Proof</u>: By the previous theorem, we have that

$$v_0 = 1 \quad , v_i = {}_0p^*_{0i} \quad (i = 1, 2, \ldots)$$

is such a solution. Thus we have to prove that there is no other solution of (1) satisfying (2). Suppose $\{a_i\}_0^\infty$ is such a solution. Then

$$a_i = \sum_{j=0}^{\infty} a_j p_{ji}$$

Multiplying through by p_{ik} and summing on i gives

$$a_k = \sum_{i=0}^{\infty} a_i p_{ik} = \sum_{i=0}^{\infty} p_{ik} \sum_{j=0}^{\infty} a_j p_{ji}$$

The repeated series on the right is convergent (to a_k) and, since it has only non-negative terms, is absolutely convergent. Interchanging the orders of summation gives

$$a_k = \sum_{j=0}^{\infty} a_j \sum_{i=0}^{\infty} p_{ji} p_{ik} = \sum_{j=0}^{\infty} a_j p^2_{jk}$$

Repeating this argument gives eventually

$$a_k = \sum_{j=0}^{\infty} a_j p^n_{ji} \qquad (n \geq 1 ; \; i = 0, 1, 2, \ldots)$$

Since the M.C. is irreducible and recurrent, for each i $\exists n \geq 1 \ni p^n_{0i} > 0$.

Thus $\qquad a_i = \sum_{j=0}^{\infty} a_j p^n_{ji} \geq a_0 p^n_{0i} > 0 \quad (i = 0, 1, \ldots ; \; n \geq 1)$

as $a_0 = 1 > 0$.

Next we introduce the quantities

$$q_{ij} = \frac{a_j}{a_i} p_{ji} \qquad (i, j = 0, 1, 2, \ldots)$$

Clearly,

$$q_{ij} \geq 0 \qquad \text{and} \quad \sum_{j=0}^{\infty} q_{ij} = \frac{1}{a_i} \sum_{j=0}^{\infty} a_j p_{ji} = 1$$

Thus we may regard the q_{ij} as the transition probabilities of some M.C. Then

$$q^2_{ij} = \sum_{k=0}^{\infty} q_{ik} q_{kj} = \sum_{k=0}^{\infty} \frac{a_k}{a_i} p_{ki} \cdot \frac{a_j}{a_k} p_{jk} = \frac{a_j}{a_i} p^2_{ji}$$

and, by repeated applications of this argument, we obtain

$$q_{ij}^{n} = \frac{q_j}{q_i} p_{ji}^{n}$$

Thus it follows that

$$\sum_{n=0}^{\infty} q_{ii}^{n} = \sum_{n=0}^{\infty} p_{ii}^{n} = \infty$$

so that the q_{ij} are transition probabilities of a recurrent irreducible M.C. From theorem 1 we have

$$\lim_{m\to\infty} \frac{\sum_{n=0}^{m} q_{io}^{n}}{\sum_{n=0}^{m} q_{oo}^{n}} = f_{io}^{*}(q) = 1$$

But by theorem 2

$$\lim_{m\to\infty} \frac{\sum_{n=0}^{m} q_{io}^{n}}{\sum_{n=0}^{m} q_{oo}^{n}} = \frac{1}{q_i} \lim_{m\to\infty} \frac{\sum_{n=0}^{m} p_{oi}^{n}}{\sum_{n=0}^{m} p_{oo}^{n}} = \frac{1}{q_i} \cdot {}_{o}p_{oi}^{*}$$

$\therefore\ 1 = {}_{o}p_{oi}^{*}/q_i$, i.e. $q_i = {}_{o}p_{oi}^{*}$. ∎

We have seen that, for a recurrent irreducible Markov chain, the sequence

$$v_0 = 1 \ , \quad v_i = {}_{o}p_{oi}^{*} \qquad (i = 1, 2, \ldots)$$

is a positive solution of the system of equations

$$(*) \qquad \sum_{j=0}^{\infty} v_j p_{ji} = v_i \qquad (i = 0, 1, 2, \ldots)$$

and that this solution is unique.

Also, for a positive recurrent irreducible Markov chain, we have seen that the positive sequence $\{\pi_i\}_0^{\infty}$ is a solution of (*) (but which may not satisfy the condition $v_0=1$). Thus, if $\mathfrak{P}$ is a positive recurrent irreducible Markov chain, we have that

$$\pi_i = c \ {}_{o}p_{oi}^{*} \quad (i = 0, 1, 2, \ldots)$$

where

$$c = \pi_0 / {}_{o}p_{oo}^{*} .$$

In considering a recurrent irreducible Markov chain with transition probabilities p_{ij} we were led to define the transition probabilities

$$(**) \qquad q_{ij} = \frac{{}_{o}p_{oj}^{*}}{{}_{o}p_{oi}^{*}} p_{ij} .$$

of some Markov chain Q which was also seen to be irreducible and recurrent.

Suppose now that $\mathfrak{P}$ (and hence Q) is positive recurrent. In this case we shall call the Markov chain Q with transition probabilities q_{ij} the <u>reversed process</u> of the chain $\mathfrak{P}$. The process Q now admits the interpretation given below. Suppose the initial distribution of the state variable is $\{\pi_i\}_0^\infty$, i.e., $Pr\{X(n) = i\} = \pi_i$ $(i = 0, 1, \ldots, \infty)$. Computing the conditional probability that the initial state was j given that the state after one transition is i yields, by Baye's rule,

$$q_{ij} = Pr\{X_0 = j \mid X_1 = i\} = \frac{Pr\{X_1 = i \mid X_0 = j\} \cdot Pr\{X_0 = j\}}{Pr\{X_1 = i\}} = \frac{\pi_j}{\pi_i} p_{ji}$$

by the stationarity of the p_{ij}. By iterating this result, we have that

$$q_{ij}^{n} = \frac{\pi_j}{\pi_i} p_{ji}^{n} \quad (n \geq 1)$$

Thus the process Q is indeed "backward in time" from the process $\mathfrak{P}$.

Let $\mathbf{P}$ be a given stochastic matrix and $u = \{u_i\}_0^\infty$ a non-negative sequence. We shall call $u = \{u_i\}_0^\infty$ a column vector. If

$Pu = u$ then u is said to be right regular relative to $\mathbf{P}$

$Pu \leq u$ then u is said to be right superregular relative to $\mathbf{P}$

$Pu \geq u$ then u is said to be right subregular relative to $\mathbf{P}$.

An r-superregular sequence $\{u_i\}_0^\infty$ is said to be minimal if $0 \leq \xi_i \leq u_i \Rightarrow \{\xi_i\}_0^\infty$ is r-regular.

<u>Theorem</u>: Let $u = \{u_i\}_0^\infty$ be r superregular wrt $\mathbf{P}$. Then

$$a_i = \lim_{n\to\infty} \sum_{j}^{\infty} p_{ij}^{n} u_j \quad (i = 0, 1, \ldots)$$

exists, and $\mathbf{a} = \{a_i\}_0^\infty$ is an r-regular vector wrt $\mathbf{P}$.

Moreover, if $b = \{b_i\}_0^\infty$ is r-regular wrt $\underline{P}$ and $b \le u$ then $b \le a$.

If we write $\quad u = a + c$

where $\quad c = u - a$

then c is minimal $\underline{P}$-superregular.

<u>Proof</u>: Since $\underline{P}u \le u$,

$$\sum_j p_{ij}^{n} u_j = \sum_j \Big(\sum_k p_{ik}^{n-1} p_{kj}\Big) u_j = \sum_k p_{ik}^{n-1} \sum_j p_{kj} u_j \le \sum_k p_{ik}^{n-1} u_k$$

i.e., $\quad P^n u \le P^{n-1} u$

so $\quad u \ge Pu \ge P^2 u \ge \dots$

and it is easy to see that $a = \lim_{n\to\infty} P^n u \le u$

We have next to show that a is an r-regular vector relative to $\underline{P}$

$$p_{ik}^{n+1} = \sum_j p_{ij} p_{jk}^{n}$$

$$\sum_k p_{ik}^{n+1} u_k = \sum_k \sum_j p_{ij} p_{jk}^{n} u_k = \sum_j p_{ij} \sum_k p_{jk}^{n} u_k$$

As $n \to \infty$, we have that

$$a_i = \lim_{n\to\infty} \sum_j p_{ij} \sum_k p_{jk}^{n} u_k$$

Formally,

$$\lim_{n\to\infty} \sum_j p_{ij} \sum_k p_{jk}^{n} u_k = \sum_j p_{ij} \lim_{n\to\infty} \sum_k p_{jk}^{n} u_k = \sum_j p_{ij} u_j$$

so that we have only to show that the interchange of the limit and summation is permitted. We have that

$$\sum_{j > N(\varepsilon)} p_{ij} u_j \le \varepsilon$$

where $N(\varepsilon)$ is large enough. Now

$$\sum_j p_{ij} \sum_k p_{jk}^{n} u_k = \sum_{j > N(\varepsilon)} p_{ij} \sum_k p_{jk}^{n} u_k + \sum_{j \le N(\varepsilon)} p_{ij} \sum_k p_{jk}^{n} u_k$$
$$\le \sum_{j \le N(\varepsilon)} p_{ij} \sum_k p_{jk}^{n} u_k + \varepsilon$$

Thus

$$a_i = \lim_{n\to\infty} \sum_j p_{ij} \sum_k p_{jk}^{n} u_k \le \sum_{j \le N(\varepsilon)} p_{ij} a_j + \varepsilon$$

and it follows then that

$$a_i = \sum_j p_{ij} a_j$$

i.e., a is r-regular relative to $\underline{P}$.

Now suppose that

$$b_i = \sum_j p_{ij} b_j \le u_i \qquad (i \geq 0)$$

Then it follows that

$$b_i = \sum_j p_{ij}^{n} b_j \le \sum_j p_{ij}^{n} u_j \quad (n \geq 1, i \geq 0)$$

$$\therefore \qquad b \le \lim_{n\to\infty} \sum_j p_{ij}^{n} u_j = a_i$$

i.e., $b \le a$.

It is trivial to prove that $c_i = u_i - a_i$ is r-superregular. It remains now only to establish that c_i is [redacted] minimal.

Suppose $\xi = \{\xi_i\}_0^{\infty}$ is r-regular relative to P and that $0 \le \xi \le c$. Then $0 \le \xi = P^n \xi \le P^n c = P^n(u-a) = P^n u - P^n a = P^n u - a$

and since $P^n u \to a$, it follows that $\xi = 0$, and this establishes the [redacted] minimal property.

Theorem: An Irreducible M.C. with transition matrix P is recurrent iff every non-negative vector v which is r-super-regular relative to P is a constant vector. (Note: By a Non-negative vector v, we mean $v_i \geq 0$ and $v_j > 0$ for some j).

Proof: Let the M.C. be recurrent and consider

$$u_i \geq \sum_j p_{ij} u_j, \quad u_i \geq 0 \quad \forall i$$

First we show that if $u_{j_0} > 0$ for some j_0, then $u_j > 0 \ \forall j$. Since the M.C. is irreducible $\exists n \geq 1 \ni p_{kj_0}^{n} > 0$. Then

$$u_k \geq \sum_j p_{kj}^{n} u_j \geq p_{kj_0}^{n} u_{j_0} > 0$$

where k is arbitrary. Now let k be fixed and set $\xi_i = u_i / u_k$. Then

$$\xi_i \geq \sum_j p_{ij} \xi_j = \sum_{j \neq k} p_{ij} \xi_j + p_{ik}$$

Iterating this inequality gives

$$\xi_i \geq \sum_{j \neq k} p_{ij} \Big[\sum_{s \neq k} p_{js} \xi_s + p_{jk}\Big] + p_{ik}$$

$$\geq \sum_{j,s \neq k} p_{ij} p_{js} \xi_s + \sum_{j \neq k} p_{ij} p_{jk} + p_{ik}$$

$$\geq \sum_{j,s \neq k} p_{ij} p_{js} \xi_s + f_{ik}^{2} + f_{ik}^{1}$$

A second iteration gives

$$\xi_i \geq \sum_{j,s,n \neq k} p_{ij} p_{js} p_{sn} \xi_n + f_{ik}^3 + f_{ik}^2 + f_{ik}^1$$

And, by induction,

$$\xi_i \geq \sum_{n=1}^{\infty} f_{ik}^n = f_{ik}^* = 1$$

since the chain is recurrent and irreducible. Thus

$$\xi_i = u_i / u_k \geq 1$$

i.e.,

$$u_i \geq u_k$$

Since i and k are arbitrary,

$$u_i = u_k \quad \forall i, k$$

To prove the converse, assume the chain is nonrecurrent and set

$$u_i = \begin{cases} f_{ik}^* & i \neq k \\ 1 & i = k \end{cases}$$

Then $u_i = f_{ik}^* = \sum_{j \neq k} p_{ij} f_{jk}^* + p_{ik} = \sum_j p_{ij} u_j \quad (i \neq k)$

and $u_k = 1 \geq f_{kk}^* = \sum_{j \neq k} p_{kj} f_{jk}^* + p_{kk} = \sum_j p_{kj} u_j$

so that u is r-superregular. Now suppose that u is a constant vector, i.e., that $u_j = f_{jk}^* = 1 \ \forall j \neq k$. Then

$$f_{kk}^* = \sum_{j \neq k} p_{kj} f_{jk}^* + p_{kk} = \sum_{j \neq k} p_{kj} + p_{kk} = \sum_j p_{kj} = 1$$

which contradicts the assumption of nonrecurrence.

__Theorem:__ For a recurrent irreducible Markov chain, the system

(1) $$v_i \geq \sum_{j=0}^{\infty} v_j p_{ji} \quad (i = 0, 1, 2, \dots)$$

where

(2) $$v_0 = 1, \ v_i \geq 0 \quad (i = 1, 2, \dots)$$

has a unique solution.

__Proof:__ We have seen that $v_i = {}_0p_i^*$ is a solution of (1) with $\geq$ replaced by = and which satisfies (2). Let

$$q_{ij} = \frac{v_j}{v_i} p_{ji} \quad (i, j = 0, 1, 2, \dots)$$

so that $q_{ij} \geq 0$ and

$$\sum_{j=0}^{\infty} q_{ij} = \frac{1}{v_i} \sum_{j=0}^{\infty} v_j p_{ji} = \frac{v_i}{v_i} = 1$$

so that $\| q_{ij} \|$ is a stochastic matrix. Moreover,

$$q_{ij}^{n} = \frac{v_j}{v_i} p_{ji}^{n}$$

and the process Q is also recurrent and irreducible.

Suppose now that $\{c_j\}_0^{\infty}$ is a solution of (1) satisfying (2). Then

$$\sum_{j=0}^{\infty} q_{ij} \frac{c_j}{v_j} = \frac{1}{v_i} \sum_{j=0}^{\infty} c_j p_{ji} \leq \frac{c_i}{v_i}$$

Thus $\{c_i / v_i\}_0^{\infty}$ is r-superregular relative to Q, and, by the previous theorem, is constant. Since $c_0 = 1 = v_0$ we conclude that $c_i = v_i$.

SUMS OF INDEPENDENT RANDOM VARIABLES AS MARKOV CHAINS

Let $x_1, x_2, \ldots$ be a sequence of integer-valued, independent, identically distributed r.v.'s and define $S_n = x_1 + x_2 + \ldots x_n$ $(n = 1, 2, \ldots)$. Take $S_0 \equiv 0$. We have seen previously that the sequence S_n determines a Markov chain. The initial state is zero since $S_0 \equiv 0$, and the state space is the collection of integers. The special feature of the Markov chain $\{S_n\}$ is its "spatial homogeneity" in that

$$p_{ij} = \Pr\{S_n = j \mid S_{n-1} = i\} = p_{0,j-i} = p_{i-j,0}$$

This property is easily seen to hold for the n-step transition probabilities also, i.e.,

$$p_{ij}^{n} = p_{0,j-i}^{n} = p_{i-j,0}^{n}$$

for,

$$p_{ij}^{2} = \sum_{k=-\infty}^{+\infty} p_{ik} p_{kj} = \sum_{k=-\infty}^{+\infty} p_{i,k+i}\, p_{k+i,j}.$$

By the spatial homogeneity of the 1-step transition probabilities,

$$p_{i,k+i} = p_{0k} \quad , \; p_{k+i,j} = p_{k,j-i}$$

hence $$p^2_{ij} = \sum_{k=-\infty}^{+\infty} p_{0k}\, p_{k,j-i} = p^2_{0,j-i}$$

The result may now be established easily by induction.

In what follows we shall assume that the Markov chain of the process $\{S_n\}$, which has he transition matrix $\| p_{ij} \|$ where $p_{ij} = \Pr\{S_n = j \mid S_{n-1} = i\}$ is irreducible, and we shall also assume that x_1 is a nondegenerate r.v., i.e., that it has at least two possible values.

We shall also have need of Green's function, defined by

$$G^n_{ij} = \sum_{m=0}^{n} p^m_{ij}$$

where, obviously,

$$G_{ij} \equiv \sum_{m=0}^{\infty} p^m_{ij} \leq +\infty$$

<u>Lemma</u>:

$$G^n_{ij} \leq G^n_{00} \quad (n = 0, 1, \dots)$$

for all i and j. In particular, as $n \to \infty$.

$$G_{ij} \leq G_{00}$$

<u>Proof</u>: We have

$$G^n_{ij} = \sum_{m=0}^{n} p^m_{ij} = \sum_{m=0}^{n} p^m_{i-j,0} = G^n_{i-j,0}$$

so that it suffices to prove

$$G^n_{i0} \leq G^n_{00}$$

for all $n \geq 0$ and all i. Now

$$G_{i0} = \sum_{m=0}^{n} p^m_{i0} = \sum_{m=0}^{n} \sum_{l=0}^{m} f^{m-l}_{i0}\, p^l_{00} = \sum_{l=0}^{n} p^l_{00} \sum_{m=l}^{n} f^{m-l}_{i0} = \sum_{l=0}^{n} p^l_{00} \sum_{r=0}^{n-l} f^r_{i0}$$

Now $$\sum_{r=0}^{n-l} f^r_{i0} \leq 1$$

so $$G^n_{i0} \leq G^n_{00} \quad . \blacksquare$$

Theorem: If $E|X_k| = E|X_1| = \sum_{j=-\infty}^{+\infty} |j| p_{0j} \quad (k=2,3,\ldots)$

and $\mu = E(X_k) = E(X_1) = \sum_{j=-\infty}^{+\infty} j p_{0j} = 0$

then the Markov chain $\{S_n\}$ is recurrent.

Remark: Since $E(X_1) = 0$ and X_1 is a nondegenerate r.v., we infer that there are positive and negative values which X_1 may achieve with positive probability. The assumption that the Markov chain $\{S_n\}$ is irreducible enables us to make use of Corollary 5.1, Chapter 2, i.e., we have to establish only the recurrence of a single state, say, the zero state.

Proof: We have by the lemma that $G_{0j}^{n} \le G_{00}^{n}$ for all j and for all $n \ge 0$. Hence,

$$\frac{1}{2M+1} \sum_{j=-M}^{+M} G_{0j}^{n} \le G_{00}^{n}$$

But

$$\sum_{j=-M}^{+M} G_{0j}^{n} = \sum_{j=-M}^{+M} \sum_{m=0}^{n} p_{0j}^{m} = \sum_{m=0}^{n} \sum_{j=-M}^{+M} p_{0j}^{m} \ge \sum_{m=0}^{n} \sum_{|j/m| \le M/n} p_{0j}^{m}$$

Hence

$$G_{00}^{n} \ge \frac{1}{2M+1} \sum_{m=0}^{n} \sum_{|j/m| \le M/n} p_{0j}^{m}$$

Since S_k is the sum of k independent identically distributed r.v. with finite mean $\mu=0$, the weak law of large numbers is applicable (Khinchine's Theorem), i.e.,

$$Pr\left\{\left|\frac{S_m - m\mu}{m}\right| \le \varepsilon\right\} = Pr\left\{\left|\frac{S_m}{m}\right| \le \varepsilon\right\} \to 1 \quad \text{as } m \to \infty$$

where $\varepsilon > 0$ is arbitrary. Now, clearly, we have

$$Pr\{|S_m| \le m\varepsilon\} = \sum_{|j| \le [m\varepsilon]} p_{0j}$$

Thus the law of large numbers (weak) may be expressed as

$$(*) \qquad H_m(\varepsilon) = Pr\{|S_m| \le m\varepsilon\} = \sum_{|j| \le [m\varepsilon]} p_{0j}^{m} \to 1 \quad \text{as } m \to \infty$$

Then, taking $M = [n\varepsilon]$ gives

$$G_{00}^{n} \ge \frac{1}{2[n\varepsilon]+1} \sum_{m=0}^{n} \sum_{|j| \le \frac{m}{n}[n\varepsilon]} p_{0j}^{m} = \frac{1}{2[n\varepsilon]+1} \sum_{m=0}^{n} \sum_{|j| \le [m\varepsilon]} p_{0j}^{m}$$

$$\ge \frac{n+1}{2[n\varepsilon]+1} \cdot \frac{1}{n+1} \sum_{m=0}^{n} H_m(\varepsilon)$$

By (*), we see that

$$\frac{1}{n+1}\sum_{m=0}^{n} H_m(\varepsilon) \to 1 \quad \text{as} \quad n \to \infty$$

Also, we have that

$$\lim_{n\to\infty} \frac{n+1}{2[n\varepsilon]+1} = \lim_{n\to\infty} \frac{n+1}{2n\varepsilon+1} = \frac{1}{2\varepsilon}$$

so that $\lim\limits_{n\to\infty} G_{00}^{n} \geq \frac{1}{2\varepsilon}$

Since $\varepsilon > 0$ may be chosen arbitrarily small, we have shown that

$$\lim_{n\to\infty} G_{00}^{n} \equiv G_{00} = \sum_{n=0}^{\infty} p_{00}^{n} = +\infty$$

i.e., the state zero, and hence the chain $\{S_n\}$, is recurrent.

<u>Theorem</u>: If

$$E|X_i| \equiv \sum_{j=-\infty}^{+\infty} |j|\, p_{0j} < \infty$$

and

$$\mu = E(X_i) = \sum_{j=-\infty}^{+\infty} j\, p_{0j} \neq 0$$

(where i = 1, 2,...) then the Markov chain $\{S_n\}$ is transient.

<u>Proof</u>: Let A_n denote the event $\{S_n = 0\}$. We recall that the condition for recurrence may be formulated as

$$\Pr\{A_n \text{ occurs } \infty \text{ often}\} = \begin{cases} 1 & \text{iff } \{S_n\} \text{ is recurrent} \\ 0 & \text{iff } \{S_n\} \text{ is transient.} \end{cases}$$

We shall make use of the strong law of large numbers:

$$\Pr\left\{\lim_{n\to\infty} \frac{S_n}{n} = \mu\right\} = 1 .$$

Consider the events

$$C_n = \left\{\left|\frac{S_n}{n} - \mu\right| > \frac{|\mu|}{2}\right\} \qquad (n = 1, 2, \ldots)$$

with C being the event that C_n occurs for ∞ many n . Any realization of the process for which $\lim\limits_{n\to\infty} \frac{S_n}{n} = \mu$ obviously does not belong to the event C. But then, by the law of large numbers, $\Pr\{C\} = 0$. Since the event A_n implies the event C_n, so $A_n \subset C_n \subset C$, we have

$$Pr\{A_n \text{ occurs } \infty \text{ often}\} \le Pr\{C\} = 0$$

from which we see that $\{S_n\}$ is transient.

Lemma: If the Markov chain $\{S_n\}$ is recurrent, then it is null recurrent.

Proof: By spatial homogeneity, we have

$$\pi_i = \lim_{n\to\infty} p_{ii}^n = \lim_{n\to\infty} p_{00}^n = \pi_0 \qquad (i = 1, 2, \ldots)$$

so that $\pi_0 > 0$ would imply $\sum_{-\infty}^{+\infty} \pi_i = \infty$, a contradiction. Therefore $\pi_i = 0 \;\; (i = 0, 1, \ldots)$

Since $\{S_n\}$ is either recurrent or null recurrent, we have that $\quad p_{0j}^n \to 0 \quad \text{as } n \to \infty$

Theorems relating to the rate of convergence of p_{0j}^n to zero are called local limit theorems. To develop some of these, we will have used of the characteristic function defined below.

Definition: If X is an integer-valued random function and $Pr\{X=k\} = p_k$ then the characteristic function of X is defined to be $\phi_X(\theta) = \sum_{\nu=-\infty}^{+\infty} p_k e^{i\nu\theta} = E[e^{iX\theta}] \quad (-\pi \le \theta < \pi)$

Note that the defining series is absolutely and uniformly convergent.

Lemma: If $X_1, X_2, \ldots, X_n$ are integer valued random functions and $S_n = X_1 + X_2 + \ldots X_n$, then $\phi_{S_n}(\theta) = \phi_{X_1}(\theta)\, \phi_{X_2}(\theta) \ldots \phi_{X_n}(\theta)$

Proof: Suppose n = 2 and that

$$Pr\{X_1 = j\} = a_j \;,\; Pr\{X_2 = j\} = b_j \quad (j = 0, \pm 1, \pm 2, \ldots).$$

Then $Pr\{X_1 + X_2 = j\} = \ldots + a_{-2} b_{j+2} + a_{-1} b_{j+1} + a_0 b_j + \ldots + a_j b_0 + a_{j+1} b_{-1} + \ldots$

so $\quad \phi_{X_1+Y_2} = \sum_{\nu=-\infty}^{+\infty} \left(\sum_{n=-\infty}^{+\infty} a_n b_{\nu-n}\right) e^{i\nu\theta} = \phi_{X_1}(\theta)\cdot\phi_{X_2}(\theta)$

The general result now follows trivially by induction. ∎

Suppose now that $X_1, \ldots, X_n$ are as before, but that they have the same distribution. Then

$$p_{0j}^{n} = \Pr\{S_n = j\}$$

in particular,

$$p_{0j} = \Pr\{X_1 = j\}$$

Then the above lemma may be used to deduce the

<u>Corollary</u>:

$$\phi_{S_n}(\theta) = [\phi_{X_1}(\theta)]^n = \sum_{\nu=-\infty}^{+\infty} p_{0\nu}^{n} e^{i\nu\theta}$$

<u>Proof</u>:

$$\sum_{\nu=-\infty}^{+\infty} p_{0\nu} e^{i\nu\theta} = E[e^{iS_n\theta}] = E[\exp\{i\theta(X_1+X_2+\cdots+X_n)\}] = \prod_{k=1}^{n} E[e^{iX_k\theta}]$$
$$= \prod_{k=1}^{n} \phi_k(\theta) = [\phi_{X_1}(\theta)]^n$$

<u>Definition</u>: We shall say that X is a periodic random variable iff $\quad \Pr\{X=\lambda\}>0 \rightarrow \lambda\in\{\omega + rc \mid r = 0, \pm1, \pm2, \ldots;\ |c| \neq 1\}$

where r, w, c are integers, w and c being fixed.

It is easy to see that, if $\{S_n\}$ is periodic, then the X_k are periodic, but that the converse does not hold gnerally.

<u>Theorem</u>: A r.v. X is periodic iff its characteristic function

$$\phi(\theta) = \sum_{\nu=-\infty}^{+\infty} p_{0\nu} e^{i\nu\theta}$$

satisfies

$$|\phi(\theta_0)| = 1$$

for some $\theta_0 \neq 0$, $-\pi \leq \theta_0 \leq \pi$

<u>Proof</u>: Suppose such a θ_0 exists. Then there is a real number $w \ni \phi(\theta_0) = e^{i\omega\theta_0}$.

Thus

$$1 = e^{-i\omega\theta_0}\phi(\theta_0) = \sum_{\nu=-\infty}^{+\infty} p_{0\nu} e^{i(\nu-\omega)\theta_0} = \sum_{\nu=-\infty}^{+\infty} p_{0\nu}\cos(\nu-\omega)\theta_0 + i\sum_{\nu=-\infty}^{+\infty} p_{0\nu}\sin(\nu-\omega)\theta_0$$

so $$1 = \sum_{\nu=-\infty}^{+\infty} p_{o\nu} \cos(\nu-\omega)\theta_0 .$$

Since $\sum_{\nu=-\infty}^{+\infty} p_{o\nu} = 1$, we must have, for those ν for which $p_{o\nu} > 0$, that $\cos(\nu-w)\theta_0 = 1$, and this implies
$$\nu = \omega + \frac{2\pi}{\theta_0} r$$
where r is any integer. Setting r=0 shows that w is an integer, and setting r=1 shows $c = \frac{2\pi}{\theta_0}$ is an integer. Obviously, $|c| \neq 1$.
From this it follows that X can attain only values of the form
$$\omega + rc \qquad (r = 0, \pm 1, \pm 2, \ldots)$$
where w, c are integers and $|c| \neq 1$ i.e. X is periodic.

Conversely, suppose the possible values of X are contained in the set $\{ \omega + rc \mid r = 0, \pm 1, \ldots \}$
where w, c are integers with $0 \neq |c| \neq 1$. Then
$$\phi(\theta) = \sum_{r=-\infty}^{+\infty} p_{o,\omega+rc} \exp[(\omega+rc)i\theta]$$
and
$$\sum_{r=-\infty}^{+\infty} p_{o,\omega+rc} = 1$$
Let $\theta_0 = 2\pi/c$, noting that $\theta_0 \neq 0$, $-\pi \le \theta_0 \le \pi$, and
$$\phi(\theta_0) = \phi\left(\tfrac{2\pi}{c}\right) = \sum_{r=-\infty}^{+\infty} p_{o,\omega+rc}\, e^{2\pi i\omega/c} e^{2\pi i r} = e^{2\pi i\omega/c} \sum_{r=-\infty}^{+\infty} p_{o,\omega+rc} = e^{2\pi i\omega/k}$$
so that
$$|\phi(\theta_0)| = \left|\phi\left(\tfrac{2\pi}{c}\right)\right| = 1$$
and this completes the proof. ∎

<u>Lemma</u>: There is a constant $\lambda > 0 \ni$
$$1 - Re[\phi(\theta)] \ge \lambda\theta^2 \qquad (-\pi \le \theta \le \pi)$$
when X is an aperiodic r.v.

<u>Proof</u>: We have
$$1 - Re[\phi(\theta)] = 1 - \sum_{j=-\infty}^{+\infty} p_{oj} \cos j\theta = \sum_{j=-\infty}^{+\infty} p_{oj}(1-\cos j\theta)$$

Now
$$1-\cos j\theta = 2\left(\sin\frac{j\theta}{2}\right)^2$$

so
$$1-Re[\phi(\theta)] = 2\sum_{j=-\infty}^{+\infty} p_{0j}\left(\sin\frac{j\theta}{2}\right)^2$$

By the Jordan inequality
$$|\sin x| \geq \frac{2|x|}{\pi} \qquad (-\pi/2 \leq x \leq \pi/2)$$

we may write
$$1-Re[\phi(\theta)] \geq \frac{2}{\pi^2}\theta^2\sum_{j=-\infty}^{+\infty} j^2 p_{0j} \quad (|j\theta| \leq \pi)$$

hence
$$1-Re[\phi(\theta)] \geq \frac{2}{\pi^2}\theta^2\sum_{j=-L}^{+L} j^2 p_{0j} \quad (|j\theta| \leq \pi)$$

But since $|j| \leq L$, the condition $|j\theta| \leq \pi$ will be met whenever
$$|\theta| \leq \frac{\pi}{L}$$

By choosing L so large that $|j| \leq L$ and $P_{0j} > 0$ for some j and taking
$$c = \frac{2}{\pi^2}\sum_{|j| \leq L} j^2 p_{0j} > 0$$

we have that
$$1-Re[\phi(\theta)] \geq c\theta^2$$

for all $|\theta| \leq \pi/L$ and $C > 0$.

We have now to consider the case $|\theta| > \pi/L$. To do so, we shall make use of the fact that X is an aperiodic r.v.

By the negation of the preceding lemma, we have that
$$|\phi(\theta_0)| = \left|\sum_{j=-\infty}^{+\infty} p_{0j} e^{ij\theta_0}\right| = 1 \qquad (-\pi \leq \theta_0 \leq \pi)$$

iff $\theta_0 = 0$. But $|\phi(\theta)| \leq 1$ is always true.
$$|1-Re[\phi(\theta)]| \geq |1-|Re[\phi(\theta)]|| = 1-|Re[\phi(\theta)]| \geq 1-|\phi(\theta)| > 0$$

for all $\theta \neq 0$, $-\pi \leq \theta \leq \pi$. In fact,
$$1-Re[\phi(\theta)] > 0$$

for all $\theta \neq 0$, $-\pi \leq \theta \leq \pi$ since $|Re[\phi(\theta)]| < 1$.

Since $1-Re[\phi(\theta)]$ is a continuous function of θ on $[-\pi, \pi]$,
$$m = \min_{\pi \geq |\theta| \geq \pi/L}\{1-Re[\phi(\theta)]\}$$

exists, and is positive (a continuous fcn on a compact set attains its extreme).

Thus we may write

$$1 - \mathcal{R}e[\phi(\theta)] > m\,\theta^2/\pi^2$$

for all $\pi \geq |\theta| \geq \pi/L$.

Taking $\lambda = \min(C, m/\pi^2)$ establishes the theorem. ∎

<u>Theorem</u>: If the r.v.'s X_k (k=0, 1, 2,...) are nonperiodic, then for some constant $A > 0$

$$p_{0k} < A/\sqrt{n}$$

for all k and all $n \geq 1$.

<u>Proof</u>: We have

$$[\phi(\theta)]^n = \sum_{\nu=-\infty}^{+\infty} p_{0\nu}^{n} e^{i\nu\theta}$$

where the series is absolutely and uniformly convergent, so

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}[\phi(\theta)]^n e^{-ik\theta}\,d\theta = \frac{1}{2\pi}\sum_{\nu=-\infty}^{+\infty} p_{0\nu}^{n}\int_{-\pi}^{+\pi} e^{i(\nu-k)\theta}\,d\theta = p_{0k}^{n}$$

In particular

$$p_{0k}^{2n} = \frac{1}{2\pi}\int_{-\pi}^{+\pi}[\phi(\theta)]^{2n} e^{-ik\theta}\,d\theta$$

so

$$p_{0k}^{2n} \leq \frac{1}{2\pi}\int_{-\pi}^{+\pi}|\phi(\theta)|^{2n}\,d\theta$$

Since the X_k are independently distributed, integer-valued r.v.'s we have

$$\phi(\theta)\,\overline{\phi(\theta)} = E(e^{iX_k\theta})\cdot E(e^{-iX_l\theta}) = E(e^{i(X_k - X_l)\theta}) = |\phi(\theta)|^2$$

so that $|\phi(\theta)|^2$ is the characteristic function of the nonperiodic, integer-valued r.v. $X_k - X_l$ $(k \neq l)$. Let $\psi(\theta) = |\phi(\theta)|^2$. Then, by the lemma above, there is a $\lambda > 0$ such that

$$1 - \psi(\theta) \leq \lambda\theta^2 \qquad (-\pi \leq \theta \leq \pi)$$

so

$$\psi(\theta) \leq 1 - \lambda\theta^2 \leq e^{-\lambda\theta^2}$$

Then

$$\int_{-\pi}^{\pi}[\psi(\theta)]^n\,d\theta \leq \int_{-\pi}^{+\pi} e^{-n\lambda\theta^2}\,d\theta = \frac{1}{\sqrt{n}}\int_{-\pi\sqrt{n}}^{+\pi\sqrt{n}} e^{-\lambda\alpha^2}\,d\alpha < \frac{1}{\sqrt{n}}\int_{-\infty}^{+\infty} e^{-\lambda\alpha^2}\,d\alpha$$

so that

$$p_{0k}^{2n} < \frac{1}{2\pi}\cdot\frac{1}{\sqrt{n}}\int_{-\infty}^{+\infty} e^{-\lambda\alpha^2}\,d\alpha = \frac{A}{\sqrt{2n}}$$

where $A = \frac{1}{\pi\sqrt{2}} \int_{-\infty}^{+\infty} e^{-\lambda\alpha^2} d\alpha$.

Since $|\phi(\theta)| < 1$ $(-\pi \le \theta \le \pi)$, we also have

$$P_{0k}^{2n+1} \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi(\theta)|^{2n+1} < \frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi(\theta)|^{2n} d\theta < \frac{1}{2\pi}\frac{1}{\sqrt{n}}\int_{-\infty}^{+\infty} e^{-\lambda\alpha^2} d\alpha = \frac{1}{\sqrt{2n}}\frac{1}{\pi\sqrt{2}}\int_{-\infty}^{+\infty} e^{-\lambda\alpha^2} d\alpha$$

so

$$P_{0k}^{2n+1} < \frac{A}{\sqrt{2n+1}}$$

References

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